Received: *October 26, 1999*, Accepted: *November 30, 1999*

Matrix theory, AdS/CFT and Higgs-Coulomb equivalence

Micha Berkooz

School of Natural Sciences, Institute for Advanced Study
Princeton, NJ 08540, USA
E-mail: berkooz@sns.ias.edu

Herman Verlinde

Joseph Henry Laboratories, Princeton University
Princeton, NJ 08544, USA
Institute for Theoretical Physics, University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam, Netherlands
E-mail: verlinde@feynman.princeton.edu

Abstract: We discuss the relation between the Matrix theory definitions of a class of decoupled theories and their AdS/CFT description in terms of the corresponding near-horizon geometry. The near horizon geometry, naively part of the Coulomb branch, is embedded in the Higgs branch via a natural change of variables. The principles of the map apply to all DLCQ descriptions in terms of hyper-Kähler quotients, such as the ADHM quantum mechanics for the D1-D5 system. We then focus the $(2,0)$ field theory, and obtain an explicit mapping from all states in the $N_0 = 1$ momentum sector of N_4 M5-branes to states in (a DLCQ version of) $AdS_7 \times S^4$. We show that, even for a single D0-brane, the space-time coordinates become non-commuting variables, suggesting an inherent non-commutativity of space-time in the presence of field strengths even for theories with gravity.

Keywords: M-Theory, p-branes, M(atrix) Theories.

Contents

1. Introduction

By now we have several non-perturbative formulations of M-theory and String theory in different backgrounds or kinematical set-ups, primarily in the frame work of Matrix theory [1] and the AdS/CFT correspondence [2]. However it is disappointing, albeit in a merry way, to go from a situation without any non-perturbative formulations to a situation with many background-dependent such formulations. One would therefore like a sequence of maps from one formulation to another. This paper is a preliminary study into the relation between the AdS/CFT correspondence and Matrix Theory in which we will suggest a derivation of the former given the latter.

More concretely, both Matrix theory and the AdS/CFT correspondence (and its generalizations) allow us to write down a description of theories which can be obtained from solitons in string theory/M-theory by decoupling gravity. The range of usefulness of the two methods is different. The advantages of the AdS/CFT duality is that it can be exhibited for a large class of field theories/gravity backgrounds, the formulation exhibits all the symmetries of the theory manifestly, and one can derive some qualitative properties of field theories from it. Its drawback is that obtaining exact quantitative results is typically possible only in extreme regimes of the parameters space of the field theory such that the gravity background is manageable, which is infrequent. The Matrix theory of decoupled theories on solitons has its limitations as well. One is restricted to a small class of theories, one looses some of the symmetries and one still needs to take the large null-momentum limit, which often makes the problem complicated. But when it exists, it exists just as well for cases where the supergravity is not weakly coupled. For example, using the DLCQ of the $(2,0)$ field theory it is easy to count precisely [3] chiral operators for every number of 5-branes but not to compute their OPE. Whereas using the AdS/CFT description it is easy to compute the 3-point function of these operators in the extreme case of a very large number of 5-branes, but not to count them precisely or compute the OPE for a finite number of 5-branes. A map between the two descriptions will perhaps enable us to enjoy the advantages of both systems, might allow us to borrow new tools from one description to be used in the other, or might teach us altogether new things.

In this paper we will suggest that it is natural to consider the AdS/CFT correspondence within the framework of Matrix theory. We emphasize that this direction is opposite to what was previously done in the literature, which is to interpret Matrix theory as a special case of an AdS/CFT-type correspondence. What we will show is that one can clearly identify the near horizon description within the DLCQ of the decoupled field theory. Thus, if one accepts these DLCQ conjectures, then the AdS/CFT correspondence is essentially a change of basis in the DLCQ Hilbert space. In this paper we will discuss some basic aspects of this identification and change of basis, and leave its more extensive elaboration to the future.

In terms of the DLCQ of the decoupled theories on the brane, the problem can be phrased in terms of the relations between the Coulomb and Higgs branches of certain (other) field theories. These field theories are those provided by Matrix theory as means of describing the dynamics on the brane prior to the decoupling from the bulk gravity; we will refer to them as the full DLCQ theories. These theories typically have a Coulomb branch, which describes gravity away from the brane, and a Higgs branch, which describes excitations inside the brane. The decoupling limit is, in the language of the full DLCQ theory, the limit in which the Higgs branch and the Coulomb branch decouple [4, 3].

The puzzle now is the following. The theory on the brane has a description in terms of gravity in the near horizon region of the soliton. Semi-classically, however, this region seems to be part of the Coulomb branch of the full DLCQ theory since the distance of the excitation from the brane is not strictly zero. Admittedly, it is the tip of the Coulomb branch where it touches the Higgs branch, but nevertheless it seems we never enter the Higgs branch. In order to reconcile these points of view, i.e. to find the near-horizon geometry in the Higgs branch, one needs to find some kind of equivalence between the tip of the Coulomb branch and the entire Higgs branch in these systems. Of course the problem is in the semiclassical statement and one can excuse oneself by appealing to large quantum fluctuations, but still one would like to do better. This is the purpose of this paper.

Once we have identified the dynamics of gravitons in the near horizon geometry in terms of dynamics on the Higgs branch, we have, in effect, derived the AdS/CFT duality from Matrix theory.

We will present in this paper only the tip of the iceberg on this map and its properties. In section 2 we present the general idea, although we will cast it a form suitable for the D0-D4 system and the D1-D5 system, which give rise to the DLCQ of the $(2,0)$ field theory [5] and the DLCQ of the little string theory [5, 4] respectively. Section 3 is a technical note on the map. In section 4 we begin focusing on the DLCQ of the $(2,0)$ field theory for a single unit of momentum along the null direction. Although in Matrix theory one is instructed to take a large number of units of such momenta, already at the single unit level we will start seeing interesting effects. Section 4 sets up the description of this model, and of what we will call "the reduced model" which is a flavor invariant version of the ADHM quantum mechanics and which will be intimately related to the near horizon geometry. Section 5 discusses the other side of the equation, i.e. the DLCQ of $AdS_7 \times S_4$ (which is the supergravity dual of the $(2,0)$ field theory). Again we specialize to the case of a single unit of momentum. Section 6 finally discusses the details of the map and some of its features. In particular we will see the significant appearance of a non-commutative[1] version of S^4.

2. The near horizon limit in matrix theory

2.1 Decoupling in low-dimensional SYM

In [5, 4] it was argued that the DLCQ of the $(2,0)$ field theory and of the "little string theory" are a $1+0$ and a $1+1$ (respectively) sigma models on certain Higgs branches (which describe instanton moduli spaces). Since both cases require a longitudinal

[1]We will use this term somewhat loosely. What we mean precisely will be made clear below.

5-brane [6], the Higgs branch in question is in both cases the dimensional reduction of 4D $\mathcal{N} = 2$ theories with $\mathrm{U}(N_0)$ gauge group, a hypermultiplet in the adjoint and hypermultiplets in a bi-fundamental of the gauge group and an $\mathrm{U}(N_4)$ flavor symmetry. We will discuss below each case at greater detail, however the basic scaling that generates the Higgs/Coulomb map is common to both and will be the topic of this section.

In our discussion here we will be schematic (we will not write all the couplings and quantum numbers).[2] Prior to decoupling, schematically, the fields in the lagrangean are shown in the table, as well as gauge fields (in the same multiplet as Y_1 and θ'), and the SYM action is

Y_1	Vector multiplets scalars
θ'	Vector multiplets fermions
H	Adjoint hypermultiplet boson
θ	Adjoint hypermultiplet fermions
Q	Bi-fundamental hypermultiplets bosons
μ	Bi-fundamental hypermultiplets fermions

$$\begin{aligned}\int d^d\sigma\, \frac{1}{g_{ym}^2} \big((\mathcal{D}Y_1)^2 + [Y_1, Y_1]^2 + \theta(\mathcal{D}\theta + [Y_1, \theta])\big) + (\mathcal{D}H)^2 + [Y_1, H]^2 + \\ +\theta'(\mathcal{D}\theta' + [Y_1, \theta']) + (\mathcal{D}Q)^2 + (Y_1 Q)^2 + \mu(\mathcal{D} + Y_1)\mu + g_{ym}^2([H, H] + Q^2)^2\,,\end{aligned} \tag{2.1}$$

where $d = 1, 2$. The dimensionality of the hypermultiplet fields changes with dimension, but the Coulomb branch coordinate Y_1 is always of dimension 1 (hence the notation Y_1).

In the analysis in [5, 4] one takes $g_{ym}^2 \to \infty$ keeping the energy fixed (or equivalently flows to the IR for fixed g_{ym}^2). If we are on the Higgs branch, then the mass of the Higgsed gauge bosons goes to infinity and they decouple, leaving us with Quantum Mechanics on the Higgs branch as the DLCQ (since we are in $0 + 1$ or $1 + 1$ dimensions, there are large fluctuations in the ground state and the Higgs branch or Coulomb branch are not really moduli spaces of the theory. Therefore decoupling is actually not automatic as we portrayed it. For the different arguments for the two cases the reader is referred to the literature).

Let us be a little more precise. In the limit $g_{ym}^2 \to \infty$ three things happen:

1. The F and D constraints are imposed exactly, giving us a non-linear sigma model.

2. If we are on the Higgs branch, and do not rescale the Higgs variables as we take $g_{ym}^2 \to \infty$ then the following happens. To keep $Y_1^2 Q^2 + Y_1\mu^2$ fixed we also do not perform any g_{ym} dependent rescaling on Y. The result is that the kinetic term of the vector multiplet vanishes. These fields become auxiliary fields and we can integrate them out. This, for example, gives the 4-Fermi interaction in the non-linear sigma model. The value of Y_1 in terms of the Higgs branch

[2] More precise statements can be found below, or in the above references.

variables is determined by the coupling of Y_1 in the lagrangean and is therefore given (qualitatively)

$$\frac{1}{2}\{Y_1, QQ^\dagger + \tilde{Q}^*\tilde{Q}^T\} + [H, [H, Y_1]] = \text{fermion bi-linear.} \tag{2.2}$$

(The details of the formula change from dimension to dimension since the number of Y_1 variables changes)

3. Finally, the physical Hilbert space is restricted to $\mathrm{U}(N_0)$ invariant wave functions.

The main point of this paper is the following. It would seem that we have lost all information about the Coulomb branch. We will argue that this is not the case. The scaling that we have performed gave a finite value to the coordinates Y_1 as a function of the Higgs branch variables. We have used this fact to integrate out the Y_1 variables. But we can also look at things in a different way and regard the Y_1's, which prior to decoupling were coordinates on the Coulomb branch, as operators on the Higgs branch. Suppose we are given a time varying quantum state on the Higgs branch $|\Psi(t)\rangle$, then by computing

$$Y(t) = \langle\Psi(t)|Y(Q, H, \mu, \theta')|\Psi(t)\rangle \tag{2.3}$$

we can transform the problem to that of an excitation moving on the Coulomb branch. The operation is nothing but a change of basis to a basis of eigenfunctions of the Y_1 operators.

What we will show is that given a Higgs branch sigma model as a DLCQ of some decoupled high dimension quantum mechanical system, then the Y_1 coordinates describe a DLCQ version of the near horizon of limit of this system. Hence the Maldacena conjecture for these cases is essentially a change of basis in Matrix theory.

This will be the main point of the paper, but it is not as straightforward as we have made it to be. The reason is that the Y_1 operators do not commute, and therefore we can not go to a basis of wave functions which are eigenfunctions of all Y_1's and describe the system as a particle moving on a new commutative space. Rather the new space will be non-commutative. We will show, however, that in the limit in which the near horizon geometry becomes flat, the non-commutativity disappears. This aspect of the near-horizon limit is similar to ideas put forward in [7], although the precise relation remains to be clarified.

The construction discussed here is also similar to that of [8]. There also coordinates analogous to the Coulomb branch reappear and become the coordinates which complete the 4 coordinates of the D3-brane to 10 coordinates of the bulk theory. There is however a difference in the sense that here we are discussing the exact quantum mechanical description in spacetime whereas the computation there

JHEP11(1999)037

applies to leading terms only in instanton computations. More closely related to the construction there might be an IKKT type conjecture [9]. The conjecture might be that the exact finite curvature $AdS_5 \times S^5$ has an IKKT-like description in terms of the large k limit of k instantons in SU(N), even for finite N. Even though this is conjecture is natural, it is somewhat disturbing. To see that, let us consider how this conjecture might be proven. The simplest way to relate the IKKT conjecture to the $D = 4, \mathcal{N} = 4$ field theory is if in the large t' Hooft coupling the full field theoretic path integral is approximated by instanton configuration, and their vicinity. This is not unlikely, as the action of other configurations is not protected and it can receive strong corrections and become infinity, leaving us only with approximate instanton configurations. The problem with this is that for small t' Hooft coupling (i.e. large curvatures) an IKKT conjecture is implausible, because in that case the path integral clearly has other contributions.

2.2 The matrix description of the $(2, 0)$ CFT

The Matrix model for the 6 dimensional $(2, 0)$ Superconformal field theory was discussed in [5],[3]. In this section we will briefly revisit it, in view of the construction from the previous section. In this case we would like to make some preliminary contact with the near horizon limit of this theory [2].

The N_0 units of momenta Matrix model for N_4 longitudinal 5-branes coupled to 11D supergravity is discussed in [6, 5]. The lagrangean of the Quantum Mechanics is

$$\begin{aligned}\int dt\ &\frac{1}{R}(\partial Y_{-1})^2 + \theta(\partial\theta) + RM_p^3\theta[Y_{-1}, \theta] + RM_p^6[Y_{-1}, Y_{-1}]^2 + \\ &+ \frac{1}{R}(\partial H)^2 + RM_p^6[Y_{-1}, H]^2 + \theta'(\partial\theta') + M_p^3\theta'[Y_{-1}, \theta'] + \\ &+ \frac{1}{R}(\partial Q)^2 + RM_p^6 Y_{-1}^2 Q^2 + RM_p^3\mu(Y_1\mu) + RM_p^6([H, H] + Q^2)^2\,. \end{aligned} \tag{2.4}$$

The lagrangeans (2.4) and (2.1) are related by setting

$$g_{ym}^2 = R^3 M_p^6 \tag{2.5}$$

and by rescaling of fields. In particular

$$Y_1 = RM_p^3 Y_{-1}\,. \tag{2.6}$$

Here Y_{-1} is the distance transverse to the brane (in the canonical metric, before we took into account the back-reaction of the brane). In the near horizon limit [2] of the $(2, 0)$ field theory we rescale this coordinate such that a dimension 2 combination

$$Y_{nh} = M_p^3 Y_{-1}$$

remains finite. We see that the relation is

$$Y_{nh} = \frac{1}{R} Y_1\,, \tag{2.7}$$

which implies that a finite Y_1 coordinate is equivalent to a finite Y_{nh}. Therefore, in the SYM decoupling limit it is precisely the near horizon coordinate which remains finite and is given as an operator on the Higgs branch (R is kept fixed throughout).

Another perspective on DLCQ and the AdS/CFT for the $(2,0)$ field theory is given in [10].

2.3 The matrix description of the "little string theory"

The Matrix description of the 6 dimensional "little string theory" [11, 12, 13] was discussed in [5] and [4]. Let us begin with the model before we go the "little string theory" decoupling limit [13]. The model is the $1+1$ dimensional model discussed above, on a cylindrical worldsheet with radius

$$\Sigma_1 = \frac{1}{R\tilde{M}_s^2}\,, \tag{2.8}$$

where $\tilde{M}_s$ is the mass scale associated with the "little string theory", and the SYM coupling is

$$\frac{1}{g_{ym}^2} = \frac{\tilde{M}_s^2}{R^2 M_p^6}\,, \tag{2.9}$$

where M_p is the 11 dimensional Planck scale which is taken to infinity, keeping $\tilde{M}_s$ and R fixed.

As before the dimension -1 Y_1 coordinate is fixed in terms of the Higgs branch variables as we go to the decoupling limit. Again, this coordinate is related to the dimension 1 coordinate by

$$Y_{-1} = \frac{Y_1}{RM_p^3}\,. \tag{2.10}$$

This is exactly the correct scaling which is required to focus on the near horizon limit of the supergravity dual of the "little string theory" [14] (i.e. the near horizon limit of the CHS vacuum [15]). Again we see that the finite Y_1 coordinate is precisely the quantity which focuses on the near-horizon limit.

We will devote the rest of this paper to the D0-D4 case. It is easy for formulate, however, the expected results for the D1-D5 as well. One expects that (in the RR sector) the Y's as functions of the Higgs branch variables will define operators that will be the coordinates of the DLCQ of the near horizon limit of the NS 5-brane — i.e. the linear dilaton background capped by $AdS_7 \times S^4$ [14]. In particular, to generate weakly coupled string perturbation theory in the linear dilaton region one needs to go to the long string in the sense of [16]. The relevant winding number will now be given by the winding of the Y_1 composite operator.

The reappearance of the Coulomb branch is particularly interesting for the D1-D5 in view of [17] (the system there is not precisely the one we are discussing here there but is clearly related). There it is the instanton moduli space on T^4 (which does not have a finite dimensional hyperkähler quotient construction) and here we are discussing the instanton moduli space on $\mathbb{R}^4$. It was shown there that associated with the singularities in the sigma model target space there is a new continuum of states in the CFT, suggesting a kind of tube. We see here that there is a natural way of regenerating this tube (although the precise relation is not clear).

We have specialized to the cases of the $(2,0)$ field theory and of the "little string theory". There are other cases, however, in which theories are given by the decoupling of Higgs and Coulomb branches, and we expect that our approach will be just as valid there. Most other examples, such as 4D $\mathcal{N}=4$, are given in terms of an impurity system (for example [18]). The procedure outlined above should work just as well, except that we will also need to use DLCQ open string field theory (this will be the general case for theories on D-branes). This issue will be discussed in [19].

3. More on the map

We have seen that the near horizon Coulomb coordinate can be thought of as an operator on the Higgs branch sigma model. This operator is determined by the equations of motion of Y. The equation of Y is of the form

$$\mathbf{L}Y^{[AB]} = \Lambda^{[AB]}\,,$$

where Λ is a fermion bi-linear and $\mathbf{L}$ is

$$\mathbf{L}Y = \frac{1}{2}\{Y, QQ^\dagger + \tilde{Q}^*\tilde{Q}^T\} + [H_n,[H_n,Y]]\,.$$

We therefore need to discuss the invertibility of $\mathbf{L}$.

Following [8, 20], it is straightforward to show that the operator is non-negative, on the space of hermitean matrices. This can be seen by the fact that

$$\mathrm{Tr}(\Omega^\dagger\mathbf{L}\Omega) = |\Omega Q|^2 + \left|\Omega\tilde{Q}^*\right|^2 + |[H_n,\Omega]|^2\,. \tag{3.1}$$

Using the equation it is also easy to see that $\mathbf{L}$ is not invertible only at singular points on the ADHM moduli space, i.e. points where parts of the $\mathrm{U}(N_0)$ is restored. The reason is that if $\mathbf{L}$ has a zero eigenvalue, then the corresponding (perhaps a few) eigenvectors Ω are hermitean matrices such that $\exp(i\Omega)$ is a subgroup of $\mathrm{U}(N_0)$ which leaves $Q, \tilde{Q}$ and H_n invariant.

Hence at generic points of the moduli space, the map is invertible. It will be interesting to study further the behavior of the map as one approaches one of the singular points. For example, some of these singularities correspond to instantons shrinking to zero size. As we approach such a point, the inverted operator $\mathbf{L}^{-1}$ will diverge which corresponds to at least one eigenvalue in the Y coordinate going to ∞. This is to be expected from the UV/IR relation [21]. Since this describes a point like object in the field theory, it will be associated with an excitation of the bulk at Y approaching ∞.

JHEP11(1999)037

3.1 The UV/IR relation and the origin of the Higgs branch

Let us elaborate somewhat the relation between (2.2) and the UV/IR relation [21]. Although the following discussion follows from conformal invariance, and thus not really a test of the map, it does demonstrate some interesting aspects of the map. A related discussion appears in [22].

The SO(5) quantum numbers of a state can only be carried by the fermion bilinear terms on the r.h.s. (since bosons on the l.h.s. do not carry SO(5) quantum numbers). Suppose we fix these quantum numbers, i.e. we fix the r.h.s. (in the sense of its action on a state). Now consider making the state smaller and smaller in 4 remaining coordinates of the brane (transverse to time and to the null circle directions). This means that the values of the l.h.s. get smaller and smaller. When we solve for Y_1, i.e. the position of the excitation in AdS_7 (with fixed the quantum numbers on the S^4), then it increases. I.e. as the object becomes more and more localized, its image in AdS_7 approaches the boundary $Y \to \infty$.

The map (2.2) in effect seems to regulate the singularity at the origin of the Higgs branch (neglecting the decoupled center of mass coordinate) since it maps it to the boundary of AdS, which is regular. Furthermore, from this point of view it is natural to have the 6D tensor multiplet of the M5-branes at the boundary of AdS, since in the Higgs branch picture one can locate the corresponding state at the origin of the Higgs branch. 'Natural' in this context means that, if we would blow up the singularity, this state mixes with all the rest [3].

4. The Higgs branch for a single D0-brane

The D0-D4 system is easier to analyze and we will focus on it. We have seen that for all N_4 and N_0 there are operators on the Higgs branch which are the coordinates on the near horizon space-time. In the remainder of this paper we will focus on the case of $N_0 = 1$. Even though the Matrix theory limit requires $N_0 \to \infty$ for fixed N_4, this simple case demonstrates some important aspects of the construction.[3]

[3]It is actually particularly interesting to better understand the map for $N_0 > 1$ in the limit that both N_0 and N_4 are taken to infinity. This limit might be useful in understanding the structure of the bound state of (only) the D0-branes. We are able to map the Higgs branch to dynamics on the Coulomb branch with arbitrarily low curvature (set by N_4). In particular, there exist states on the Higgs branch which correspond to the bound states of D0-branes in the near horizon Coulomb branch. This map, therefore, maps a rather complicated non-abelian problem to a problem in a sigma model with no non-abelian gauge dynamics. For example, proving the existence of these states on the Higgs branch is simpler then proving their existence in flat space Matrix theory [23]. In fact, the counting can be easily done for any N_4 and N_0 [24] (and adapted for Matrix theory purposes in [3]). As we take $N_0 \to \infty$ the bound state on the near horizon Coulomb branch expands in the Coulomb branch coordinates. To approach the flat space limit we also need to take $N_4 \to \infty$, such that the size of the bound state is always smaller then the radius of curvature of AdS. In this case we can regard the wave functions on the sigma model as a regulated version of the bound state in flat space, and we can remove the regulator at will.

4.1 Quantum mechanics on the ADHM moduli space

The field content of the ADHM moduli space for $N_0 = 1$ and arbitrary N_4 is shown in the table.

	$\mathrm{U}(1)_{\mathrm{gauge}}$	$\mathrm{SU}(2)_R$	$\mathrm{SU}(2)_L$	Spin(5)	$\mathrm{U}(N_4)$
H	0	**2**	**2**	**1**	**1**
Θ	0	**1**	**2**	**4**	**1**
w	**1**	**2**	**1**	**1**	$\mathbf{N_4}$
μ	**1**	**1**	**1**	**4**	$\mathbf{N_4}$

The $H - \Theta$ multiplet is free, and for the most part we wil ignore it (an identical sector will appear in the near horizon Coulomb branch and the identification between them is immediate). Generally, $\alpha, \beta, \ldots$ will denote $\mathrm{SU}(2)_R$, $i, j, \ldots$ will denote $\mathrm{U}(N_4)$ indices and $A, B, \ldots$ will denote spinor Spin(5) indices (w is what we called Q before).

The hypermultiplets w satisfy D-term constraints

$$(\sigma^a)^\alpha_\beta w^j_\alpha \bar{w}^\beta_j = 0\,, \qquad a = 1, \ldots, 3 \tag{4.1}$$

and j is the $\mathrm{U}(N_4)$ index (whenever possible we will use the notation of [8]). These constraints imply that $w^j_\alpha \bar{w}^\beta_j$ is proportional to the identity 2×2 matrix, and it is convenient to define

$$Q^2 = \frac{1}{2}\,\mathrm{Tr}(w^j_\alpha \bar{w}^\beta_j)\,. \tag{4.2}$$

The fermions are also restricted by the relation

$$\bar{\mu}^A_j w^j_\alpha + J^{AB}\epsilon_{\alpha\beta}\mu^j_B \bar{w}^\beta_j = 0. \tag{4.3}$$

This equation comes from integrating out the superpartners of the Coulomb branch coordinates (J^{AB} denotes the antisymmetric form of USp(2)). In addition we need to mod out by $\mathrm{U}(1)_{\mathrm{gauge}}$.

Although we will discuss in a moment the solution to these equations when we fix the $\mathrm{U}(N_4)$ symmetry, let us first proceed more generally. As in [8], it is convenient to classify the solutions to the linear constraint (4.3) in the following way. One class of solutions, which we will denote by ν^B_k, $k = 1, \ldots, N_4 - 2$, satisfies that μ and $\bar{w}$ are orthogonal (in flavor indices, for every A and α). The second class of solutions is given by

$$\begin{aligned} \mu^j_B &= \frac{1}{Q} w^j_\alpha \eta^\alpha_B\,, \\ \bar{\mu}^B_j &= \frac{1}{Q} \bar{\eta}^B_\alpha \bar{w}^\alpha_j\,, \end{aligned} \tag{4.4}$$

where the η satisfy the pseudo-reality condition

$$\bar{\eta}^A_\alpha + J^{AB}\epsilon_{\alpha\beta}\eta^\beta_B = 0\,. \tag{4.5}$$

Regarding this Higgs branch as the moduli space of instantons these are nothing but the superconformal zero modes.

4.2 The reduced model

We are interested in mapping states in this quantum mechanics to states on the near horizon Coulomb branch. There is, however, an obstacle which is that many states on the Higgs branch carry $\mathrm{U}(N_4)$ quantum numbers, which clearly does not exist on the Coulomb branch side. We will therefore restrict ourselves to states that are invariant under $\mathrm{U}(N_4)$. This is compatible with (2.2), which is also $\mathrm{U}(N_4)$ invariant.[4]

To discuss states that are invariant under $\mathrm{U}(N_4)$ (and under $\mathrm{U}(1)_{\text{gauge}}$), it is convenient to define a "reduced" quantum mechanics, which is the quantum mechanics on a Hilbert space of flavor invariant states (as well as gauge invariant, of course). Actually we will fix the gauge only partially and work with a larger Hilbert space, to which we will also refer at times as the "reduced" Hilbert space. The procedure that we will adopt is to gauge fix the bosonic part of the hypermultiplets. A convenient choice of gauge will be

$$w_\alpha^j = \begin{pmatrix} Q & 0 & 0 & \dots & 0 \\ 0 & Q & 0 & \dots & 0 \end{pmatrix}. \tag{4.6}$$

It is clear that gauge and flavor invariant functions can be thought of as functions of a single variable Q, but with as many fermions as we had before, i.e. this space has a single bosonic coordinate, 8 fermionic coordinates η and $4(N_4-2)$ complex pairs of fermions $\nu_k^A, \bar{\nu}_A^k$. We have clearly not fixed the gauge completely and, for example, still have $\mathrm{U}(N_4-2)$ flavor symmetry acting on the fermions which we will now eliminate

The commutation relation on the remaining fermions can still be taken to be the canonical ones:

$$\{\eta_A^\alpha, \eta_B^\beta\} = \epsilon^{\alpha\beta} J_{AB}\,, \qquad \{\nu_k^A, \bar{\nu}_B^{k'}\} = \delta_{k'}^k \delta_B^A\,. \tag{4.7}$$

The reduced Hilbert space. We start with a model that has $\mathrm{SU}(N_4)_{\text{global}} \times \mathrm{U}(1)_{\text{gauge}}$ symmetry. The $\mathrm{U}(1)_{\text{gauge}}$ is the diagonal of the $\mathrm{U}(N_4)$ that acts on the hypermultiplets. After we go to the special gauge above the remaining symmetry is $\mathrm{SU}(N_4-2)_{\text{flavor}} \times \mathrm{U}(1)_{\text{gauge}}$. We are interested in states which are invariant under all these symmetries. Clearly Q and η are invariant under these symmetries, hence all the restrictions will be in the $\nu - \bar{\nu}$ Fock space. There are two states in this Fock space, $|+\rangle$ and $|-\rangle$, which satisfy

$$\nu_k^A |-\rangle = 0\,, \qquad \bar{\nu}_A^k |+\rangle = 0\,. \tag{4.8}$$

These states are exchanged under $\nu \leftrightarrow \bar{\nu}$, hence their natural $\mathrm{U}(1)$ charge assignment will be opposite. If we set the $\mathrm{U}(1)$ gauge charge of ν to 1, then $|+\rangle$ has charge $-2(N_4-2)$. The requirement of gauge invariance then tells us that we are restricted to states of the form $\nu^{2(N_4-2)}|+\rangle$. Furthermore, since all the indices on the ν operators are that of a fundamental index, we can only contract them by a baryonic vertex.

[4] The interpretation of states that are charged under this group is not clear.

The Hilbert space of gauge invariant and flavor invariant functions is therefore of the form

$$f(Q,\eta)\epsilon^{l_1\cdots l_{N_4-2}}\epsilon^{t_1\ldots t_{N_4-2}}\nu^{A_1}_{l_1}\cdots\nu^{A_{N_4-2}}_{l_{N_4-2}}\nu^{B_1}_{t_1}\cdots\nu^{B_{N_4-2}}_{t_{N_4-2}}|+\rangle\,; \tag{4.9}$$

for brevity we will denote N_4-2 by N.

Aspects of the reduced supercharges and hamiltonian. We have seen that the structure of the Hilbert space is simple enough. One would like to know whether "reducing" in this way has not made the dynamics, as encoded by the hamiltonian or supercharges, too complicated. Fortunately, The price that we pay for reducing is minimal. In the full ADHM quantum mechanics flavor invariance is the equation

$$(T^a_{bos}+T^a_{ferm})\Psi(w,\bar w,\mu,\bar\mu)=0\,, \tag{4.10}$$

where a is an $u(N)$ index, T^a_{bos} is the action on the bosons and T^a_{fer} is the action on the fermions. (4.10) equates derivatives with respect to $w,\bar w$ (along orbits of the flavor symmetry) with fermionic bi-linear operators acting on the states. We can now replace the bosonic derivatives in these directions with fermion bi-linear operators. The only bosonic derivative that will be left is in the Q direction. We have, however, generated new fermion tri-linear terms, but since the original supercharges also contained tri-linear fermion terms, this does not complicate the system (A toy example of this reduction procedure is discussed in appendix A).

Given the supercharges of the initial ADHM quantum mechanics we can determine the supercharges of the reduced system. Alternatively since we know that the supercharges have at most tri-linear fermion terms we can compute them explicitly by requiring closure of the supersymmetry algebra. The supercharges are computed in this way in appendix B, and the result is

$$Q^\alpha_A=\eta^\alpha_A\Big(\frac{\partial}{\partial Q}+\frac{a}{Q}\Big)+\frac{2}{Q}\eta^\alpha_B L^{BC}J_{CA}+\frac{1}{Q}M^{\alpha\beta}\eta^\gamma_A\epsilon_{\beta\gamma}\,. \tag{4.11}$$

where L is the SO(5) generator in the $\nu-\bar\nu$ sector, M is the $\mathrm{SU}(2)_R$ generator on η (for precise conventions see appendix B), and a is a specific constant which we have not determined since will we not require it.

5. DLCQ of the Poincaré patch of $AdS_7\times S^4$

We would like to perform a DLCQ quantization of M-theory on $AdS_7\times S^4$. The correct and full DLCQ of this background is of course the quantum mechanics on the ADHM moduli space. We would like, however, to start with this spacetime and try and write an approximate DLCQ description of that. This step, however, is somewhat problematic. We can not try and write a DLCQ for the entire full cover, because this space does not have any null isometries. However, the Poincare patch has a null isometry. If we use the coordinates U, X^i, $i = 0, \ldots, 5$ then we can mod out by the symmetry $x^+ \to x^+ + R$. The problem now is that, since there is a fixed point, the quotient is singular at $U = 0$. Away from the singularity we can write a DLCQ, which will be a finite dimensional quantum mechanics. As we approach the fixed point of the null translation, the quantum mechanics will become singular. This may invalidate the whole approach but we will argue that this is not the case. Having a singularity in the hamiltonian of the DLCQ is not a source for concern as long as one asks the right questions. There is, for example, no point in asking what is the ground state of the quantum mechanics, but if the hamiltonian is a good differential operator on functions supported away from the singularity, then it is a sensible question to ask what is the dynamics of a wave packet there.

We will restrict our attention to the DLCQ for $N_0 = 1$, i.e. for a single unit of momentum along the null circle. In this case the model is a quantum mechanical sigma model, which describes the motion of the gravity multiplet on this background. Taking the metric of $AdS_7 \times S^4$ to be[5]

$$r dx^2 + \left(\frac{dr}{r}\right)^2 + d\theta^2$$

the equation of motion of a scalar particle of mass m is

$$\left(\partial_x^2 + \frac{1}{r}\partial_r r^4 \partial_r + r\partial_\theta^2 + rm^2\right)\Psi = 0\,.$$

Going to DLCQ

$$H \sim \frac{1}{P_+}\left(\partial_{x_\perp}^2 + \frac{1}{r}\partial_r r^4 \partial_r + r\partial_\theta^2 + rm^2\right),$$

which is a the hamiltonian on a sigma model with metric

$$dx^2 + \frac{1}{r^3}(dr^2 + r^2 d\theta^2)$$

up to a shift in the mass and an r-dependent rescaling on Ψ. Not surprisingly, this is the same as the metric that a D0 sees near a D4 brane, when we rescale the coordinates as we go to the M5-near horizon limit.

The full lagrangean of a D0 away from a D4 brane was determined, to lowest order in derivative expansion, in [25]. In addition to a decoupled $\mathbb{R}^4$ (associated with the hypermultiplet H which is decoupled for $N_0 = 1$) there are 5 coordinates U^i ($U^2 = \sum {U^i}^2$) which[6] parameterize 5 coordinates transverse to the brane, i.e one

[5] We use the coordinate r which is proportional to the distance from the brane in the uncorrected metric

[6] Generally the letter U will denote commuting coordinates on the Coulomb branch. Y^i denotes the non-commuting ones.

radial coordinate and the four of S^4. The most general form of an SO(5) invariant lagrangean is given in [25, equation (3.10)], and it is:

$$-f(U)\left(\dot{U}^j\dot{U}^j+i(\bar{\rho}\dot{\rho}+\rho\dot{\bar{\rho}})\right)+\dot{U}^i f_{,j}(\rho\gamma^{ij}\bar{\rho})+\frac{1}{2}(f_{ij}-f^{-1}f_{,i}f_{,j})\left(\rho\gamma^i\bar{\rho}\rho\gamma^i\bar{\rho}+\rho\gamma^i\rho\bar{\rho}\gamma^i\bar{\rho}\right), \tag{5.1}$$

where ρ (which is denoted in [25] by η) is a **4** of USp(2) and ρ and $\bar{\rho}$ together form a doublet of $SU(2)_R$. The most general $f(U)$ allowed by the (super)symmetries is $f = c_0 + c_1/U^3$. This is the metric that a D0-brane sees near a D4-brane (for proper $c_{0,1}$). When we take $M_p \to \infty$, and properly normalize the U coordinates as to go to the near horizon limit, we obtain that the function f is given by

$$f(U) = \frac{N_4}{RM_p^3U^3}\,. \tag{5.2}$$

In section 6.5 we will partially match terms in the supercharges for this system with terms in the supercharges of the ADHM sigma model. The term which is most interesting to match is the one that contains derivatives with respect to U^i because transforming this term into a Higgs branch expression will rely most heavily on the maps from the Higgs to the Coulomb branch. The ∂_U term appears in the Coulomb branch supercharges as

$$Q^\alpha_A = U^{\frac{3}{2}}\rho^\alpha_B J^{BC}\gamma^i_{AC}\frac{\partial}{\partial U^i} + 3\ \rho\ \text{terms}\,. \tag{5.3}$$

One more point is in order. We have restricted ourselves to $N_0 = 1$. One can ask whether there is a generalization of this formula to larger values of N_0. There are several ways to try and obtain approximate answers but, since there is no established supersymmetric non-renormalizaton theorem for these cases, we will not explore this generalization here. One thing is clear, however: when the quanta that carry DLCQ momenta are close to each other (compared to the scale set by the curvature of space-time) we should approximately obtain the $\mathcal{N} = 16$ BFSS lagrangean as a limit of the ADHM sigma model.

6. Higgs Coulomb equivalence

6.1 The coordinate algebra

The relation (2.2) defines a set of operators $Y^{[AB]}$ which are coordinates on the near horizon Coulomb branch as operators on the Higgs branch. This relation is valid for all N_0 and N_4. For the case $N_0 = 1$ this relation simplifies significantly and becomes

$$Y^{[AB]} = \frac{\nu^A_k\bar{\nu}^k_F J^{BF} - \nu^B_k\bar{\nu}^k_C J^{AC} + \frac{1}{2}J^{AB}\nu^F_k\bar{\nu}^k_F}{Q^2}\,, \tag{6.1}$$

where the sum over k is $k = 1, \ldots, N$ ($N = N_4 - 2$). Using this formula we would like to study in greater detail how the correspondence works.

The equation (6.1) defines the operators Y in terms of Q^2 and in terms of the fundamental fermions. In the context of $AdS_7 \times S^4$ we should think of the 5 Y coordinates as $R^+ \times S^4$, where the R^+, which we will denote as $\bar{U}$, is associated with the additional coordinate of AdS_7. We will see below that

$$\bar{U} \sim \frac{\sqrt{2}(N)}{Q^2}\,, \tag{6.2}$$

and the S^4 manifold (which becomes fuzzy) is related to the fermion bi-linear. We therefore begin by focusing on the fermion bi-linear terms.

Before doing so, it is useful to list the main objects that we will be dealing with, which is the following "algebra of coordinates", and some of the relations between them:

1. The flavor invariant coordinates:
$$B^{[AB]} = \nu_k^A \bar{\nu}_F^k J^{BF} - \nu_k^B \bar{\nu}_F^k J^{AF} + \frac{1}{2} J^{AB} \nu_k^F \bar{\nu}_F^k\,. \tag{6.3}$$

2. The flavor charged coordinates:
$$B^{[AB]}_{l_1 l_2} = \nu_{l_1}^A \nu_{l_2}^B - \nu_{l_1}^B \nu_{l_2}^A + \frac{1}{2} J^{[AB]} J_{CD} \nu_{l_1}^C \nu_{l_2}^D\,. \tag{6.4}$$

3. The null coordinate
$$B^0_{l_1 l_2} = J_{AB} \nu_{l_1}^A \nu_{l_2}^B\,. \tag{6.5}$$

Here J is the antisymmetric forms of USp(2) and we will at times denote $B^{AB}_{l_1 l_2}$ and $B^0_{l_1 l_2}$ by $B^{AB}_{\ldots}$ and $B^0_{\ldots}$ respectively.

The operators B^{AB} and $B^{AB}_{\ldots}$ are a **5** of USp(2), and $B^0_{\ldots}$ is a singlet. The B^{AB} operators satisfy the following hermiticity relation

$$\left(B^{AB}\right)^{\dagger}{}_{AB} = B^{\dagger}{}_{BA} = J_{AA'} B^{A'B'} J_{B'B} \tag{6.6}$$

and are therefore related, as expected, to 5 real coordinates. The relation is

$$B^{13} = B^1 + iB^2 = -B^{42\,\dagger}\,, \qquad B^{14} = B^3 + iB^4 = B^{32\,\dagger}\,, \qquad B^{12} = B^5 = -B^{34}\,.$$
$$J_{AC} J_{BD} B^{AB} B^{CD} = 4 \textstyle\sum_{i=1}^{5} B^{i\,2}\,. \tag{6.7}$$

Some of the relations between these operators, which will be useful below, are:

$$[B^{[AB]}, B^{[CD]}] = J^{AC} L^{BD} - J^{AD} L^{BC} - J^{BC} L^{AD} + J^{BD} L^{AC}\,. \tag{6.8}$$
$$[B^{[AB]}, B^0_{l_1 l_2}] = 2 B^{[AB]}_{l_1 l_2} \tag{6.9}$$
$$[B^{[AB]}, B^{[CD]}_{l_1 l_2}] = \left(J^{AC} J^{BD} - J^{AD} J^{BC} - \frac{1}{2} J^{AB} J^{CD}\right) B^0_{l_1 l_2} \tag{6.10}$$
$$\begin{aligned} J_{AC} J_{BD} B^{[AB]}_{l_1 k_1} B^{[CD]}_{l_2 k_2} &= \Big(\frac{1}{2} B^0_{l_1 k_1} B^0_{l_2 k_2} - (l_1 \leftrightarrow l_2) - \\ &\quad - (k_1 \leftrightarrow k_2) + (k_1, l_1 \leftrightarrow k_2, l_2)\Big) + \cdots\,, \end{aligned} \tag{6.11}$$

where $\cdots$ are terms which are symmetric under $l_1 \to l_2$ or $k_1 \to k_2$ and, as will be clear below, will not play a role in our analysis. These operators and the relations between them are the structures which organize the correspondence for the $N_0 = 1$ case.

Note equation (6.8) which encodes the non-commutativity of the coordinates. The non-commutativity is closely related to the truncation of the spectrum of KK states on S^4, and since in the ADHM quantum mechanics the spectrum of chiral operators truncates correctly for every N_4 and N_0 we expect that this non-commutativity will persist (in some form) even in this limit. This implies that spacetime becomes "non-commutative" even for closed strings in the presence of a closed string field strength. The length scale associated with this non-commutativity may be much smaller than the string scale, but since D-branes can probe sub-stringy structures, they can still resolve it, which is another interpretation of our results for the D0-D4 system.

6.2 Mapping of states

The Fermi surface. The purpose of this section is to show how the operators of the coordinate algebra organize the states of the reduced system. We will show that the 5 B^{AB} operators organize these states in the same way that the 5 commutative coordinates of R^5 organize the spherical harmonics on the unit sphere.

The class of relevant states is given in (4.9). Each baryonic contraction $\epsilon^{\cdots}\nu^{A_1}_{\cdot}\cdots\nu^{A_N}_{\cdot}$ is the N-th tensor symmetric representation of the $\mathbf{4}$ of USp(2). Therefore all the states are in the product of two such representations. In particular there is an USp(2) singlet state given by

$$|\phi\rangle = \epsilon^{l_1\cdots l_N}\epsilon^{t_1\cdots t_N}B^0_{l_1t_1}\cdots B^0_{l_Nt_N}|+\rangle\,. \tag{6.12}$$

This state corresponds to the lowest spherical harmonic, i.e. the constant function on the sphere.

We will refer to this state as the "Fermi surface". The reason is that all higher spherical harmonics will be given as excitations of this state, in a similar way to exciting a fermion from below a Fermi surface to a level above it.

Higher spherical harmonics. The rest of the spherical harmonics on S^4 match with states in the Fock space by the following correspondence. Suppose we are given a j-th symmetric traceless tensor of SO(5) which we will write in USp(2) conventions as $V_{[A_1B_1]\ldots[A_jB_j]}$, then the map (6.1) implies

$$\begin{aligned}V_{[A_1B_1]\cdots[A_jB_j]}U^{[A_1B_1]}\cdots U^{[A_jB_j]} &\leftrightarrow V_{[A_1B_1]\cdots[A_jB_j]}B^{[A_1B_1]}\cdots B^{[A_jB_j]}|\Phi\rangle = \\ &= 2^j\frac{N!}{(N-j)!}V_{[A_1B_1]\cdots[A_jB_j]}\epsilon^{\cdots}\epsilon^{\cdots}B^{A_1B_1}_{\cdots}\cdots B^{A_jB_j}_{\cdots}B^0_{\cdots}\cdots B^0_{\cdots}|+\rangle\,.\end{aligned} \tag{6.13}$$

We have restricted the V's to live in a single irreducible representation. If we allow general V then the relation is still correct as long as we do not use on the l.h.s. the relation $U^2 = \text{const.}$, but leave it as an operator U^2. The reason is that B^2 evaluated on the different wave function on the r.h.s. is not a constant (although for large N the corrections are roughly suppressed by powers of $\frac{j}{N}$ — this will be discussed further in section 6.4). If we would allow a general V above and use $U^2 = \text{const.}$ then each representation would be over-defined (by all V's with a larger number of vector indices), and the inequality of B^2 on the different representations would make the definitions incompatible.

Counting of states. We would like to show that all the states (4.9) are in $1-1$ correspondence with spherical harmonics, and that the spectrum truncates at the correct place. The last statement is easy to verify and should comes as no surprise since the ADHM quantum mechanics gives the correct spectrum of chiral operators [3]. The state with the largest number of flavor charged coordinates has N such coordinates

$$\epsilon^{\cdots}\epsilon^{\cdots}B^{A_1B_1}_{\cdots}\cdots B^{A_NB_N}_{\cdots}|+\rangle$$

so there are overall N_4-1 spherical harmonics functions in the spectrum. This exactly matches the expected truncation of states on the S^4 from the $(2,0)$ CFT point of view (a truncation similar to the truncation to $\mathrm{Tr}\, X^2, \ldots, \mathrm{Tr}\, X^{N_3}$ for $AdS_5 \times S^5$).

Supergravity is, in a sense, given by the quantum mechanics of the system (5.1). Indeed there we do not see any truncation of the spectrum. However, the full non-perturbative formulation, which is the ADHM QM, includes the correct cut-off.

Next we would like to show that all the states in (4.9) are generated by acting with the coordinate algebra on the Fermi surface. We regard the states in (4.9) as living in the product of two N-th symmetric tensor products of **4**'s of USp(2). To examine what representation appears in this product, it's enough to choose a fixed vector of our choice in one of the representations. We will chose this state to be such that all the USp(2) indices are "1". The relevant class of states is therefore

$$\epsilon^{l_1\cdots l_N}\epsilon^{t_1\cdots t_N}\nu^1_{l_1}\cdots\nu^1_{l_N}\nu^2_{t_1}\cdots\nu^2_{t_{n_2}}\nu^3_{t_{n_2+1}}\cdots\nu^3_{t_{n_2+n_3}}\nu^4_{t_{n_2+n_3+1}}\cdots\nu^4_{t_N}|+\rangle\,.$$

It is clear, however, that this state equals

$$\epsilon^{l_1\cdots l_N}\epsilon^{t_1\cdots t_N}B^{12}_{l_1t_1}\cdots B^{12}_{l_{n_2}t_{n_2}}B^{13}_{l_{n_2+1}t_{n_2+1}}\cdots B^{13}_{l_{n_2n_3}t_{n_2+n_3}}B^{14}_{l_{n_2+n_3+1}t_{n_2+n_3+1}}\cdots B^{14}_{l_Nt_N}|+\rangle\,.$$

Hence all the states in (4.9) are generated by the coordinate algebra.

Now that we have identified states which correspond to the spherical harmonics and operators which can play the role of coordinates, we can examine the correspondence more carefully. We would now like to address the questions

1. Is the map (6.13) unitary ?
2. What is the size of the sphere ?
3. To what extent are the B's coordinates on a sphere ?

The answer that we will obtain is that for low lying Kaluza-Klein spherical harmonics, the map is unitary, the states correspond to spherical harmonics on a sphere of radius $\sqrt{2}N$ and the B's act as coordinates. Fortunately, low-lying Kaluza-Klein states for our purposes certainly include all states with wavelength larger the Planck scale.

6.3 The dilute gas approximation

This section will touch upon some aspects of the coordinate algebra in the "dilute gas approximation". This approximation is the leading term for $j \ll N$ where j is the level of the spherical harmonics (we will show that the leading correction is actually j/N^{α} where $\alpha \sim O(1)$, $\alpha < 1$ will be determined in the next subsection).

The "dilute gas approximation" is the following. For every j the state is given by

$$\epsilon^{l_1 \dots l_N} \epsilon^{t_1 \dots t_N} B^{A_1 B_1}_{l_1 t_1} \cdots B^{A_j B_j}_{l_j t_j} B^0_{l_{j+1} t_{j+1}} \dots B^0_{l_N t_N} |+\rangle \,.$$

In the limit $j \ll N$ most of the B's are B^0. In this approximation whenever a computation receives a contribution both from B^{AB}_{lt} and from the B^0_{lt} we will neglect the former, since the contribution from such term will be proportional to j vs. contributions from the 2nd term which will be proportional to $N - j$.

Within this approximation we lose the non-commutativity in the system, and hence the following things happen:

1. the map

$$\begin{aligned} V_{i_1 \dots i_j} U^{i_1} \cdots U^{i_j} \leftrightarrow V_{i_1 \dots i_j} B^{i_1} \cdots B^{i_j} |\Phi\rangle \leftrightarrow \\ \leftrightarrow (2N)^j V_{i_1 \dots i_j} \epsilon^{l_1 \dots l_N} \epsilon^{t_1 \dots t_N} B^{i_1}_{l_1 t_1} \cdots B^{i_j}_{l_j t_j} B^0_{l_{j+1} t_{j+1}} \dots B^0_{l_N t_N} |+\rangle \end{aligned} \tag{6.14}$$

   becomes exact even for V's which are not irreducible.

2. The operators B^{AB} are now commuting. The action of B^{AB} on the states is now

$$B^{AB} |(A_1 B_1) \dots (A_j B_j)\rangle \sim 2N |(AB)(A_1 B_1)(A_2 B_2) \dots (A_j B_j)\rangle$$

   and correspondingly

$$[B^{AB}, B^{CD}] |(A_1 B_1) \dots (A_j B_j)\rangle = 0 \,.$$

3. The states move on a sphere of radius $\sqrt{2}N$.

$$\begin{aligned} J_{AA'}J_{BB'}B^{AB}B^{A'B'}|(A_1B_1)\cdots(A_jB_j)\rangle &\sim \\ &\sim N^2 J_{AA'}J_{BB'}|(A_1B_1)(A_2B_2)\cdots(A_jB_j)(AB)(A'B')\rangle \\ &= 8N^2|(A_1B_1)(A_2B_2)\cdots(A_jB_j)\rangle\,, \end{aligned} \tag{6.15}$$

which implies a fixed radius of the sphere

$$\sum_i B^{i2} = 2N^2 \tag{6.16}$$

(this is related to but not precisely the radius of the sphere as seen in supergravity. To compare with that one needs to compute the hamiltonian or the supercharges. Although we will briefly discuss the supercharges below, we will not touch upon this point).

6.4 An exact computation

To examine the departures from the dilute gas approximation we would like to perform an exact computation of things such as the norms, $\langle B^2\rangle$ etc. It will be sufficient to do so in a single state in each irreducible representation. The states that are the most convenient to use are the states

$$|n\rangle = B^{13^n}|\phi\rangle\,, \qquad \langle n| = \langle\phi|B^{24^n}\,.$$

As mentioned above, in a 5-vector notation $B^{13} = B^1 + iB^2 = B^{24*}$. Therefore B^{13^n} will generate a state already in the symmetric, traceless rep. of SO(5).

We would first like to evaluate corrections to the norms of the states, vs. the norms of a state moving on a commuting S^4. On the classical geometry side (on the unit sphere) (we will use the coordinates U here to denote ordinary commuting coordinates on the unit sphere)

$$\|U^{13^n}\|^2 \propto \frac{\Gamma(n+1)}{\Gamma(n+5/2)}$$

or in a form which will be more useful

$$\frac{\|U^{13^n}\|^2}{\|U^{13^{n-1}}\|^2} = \frac{n}{n+3/2}\,,$$

where $\|U^{13^n}\|^2 = \int d\Omega |U^{13}|^{2n}$, whereas on the quantum non-commutative geometry side

$$\begin{aligned} \langle n|n\rangle &= \langle n-1|B^{24}|n\rangle \\ &= \langle n-1|n(N-n+1)|n-1\rangle + 2(N-n)\frac{2^n N!}{(N-n)!}\times \\ &\quad\times\langle n-1|\epsilon^{l_1\cdots l_N}\epsilon^{t_1\cdots t_N}B^{13}_{l_1t_1}\cdots B^{13}_{l_nt_n}B^{24}_{l_{n+1}t_{n+1}}B^0_{l_{n+2}t_{n+2}}\cdots B^0_{l_Nt_N}|+\rangle\,. \end{aligned} \tag{6.17}$$

The overlap in the last line is not zero and is partially determined by quantum numbers

$$\begin{aligned}
&\langle n-1|\epsilon^{l_1\dots l_N}\epsilon^{t_1\dots t_N}B^{13}_{l_1t_1}\cdots B^{13}_{l_nt_n}B^{24}_{l_{n+1}t_{n+1}}B^{0}_{l_{n+2}t_{n+2}}\cdots B^{0}_{l_Nt_N}|+\rangle = \\
&\quad = \frac{n}{n+\frac{3}{2}}\langle n-1|\ \frac{1}{4}J_{CC'}J_{DD'}\epsilon^{l_1\dots l_N}\epsilon^{t_1\dots t_N}B^{13}_{l_1t_1}\cdots B^{CD}_{l_nt_n}B^{C'D'}_{l_{n+1}t_{n+1}}B^{0}_{l_{n+2}t_{n+2}}\cdots B^{0}_{l_Nt_N}|+\rangle \\
&\quad = \frac{n}{n+\frac{3}{2}}\cdot\frac{1}{2}\cdot\epsilon^{l_1\dots l_N}\epsilon^{t_1\dots t_N}B^{13}_{l_1t_1}\cdots B^{12}_{l_{n-1}t_{n-1}}B^{0}_{l_nt_n}\cdots B^{0}_{l_Nt_N}|+\rangle\,. \qquad (6.18)
\end{aligned}$$

Hence the recursion relation is

$$\langle n|n\rangle = 2N^2\frac{n}{n+\frac{3}{2}}\left(1+O\left(\frac{n^2}{N}\right)\right)\langle n-1|n-1\rangle\,.$$

As before the leading size of the sphere is $\sum_i {B^i}^2 = 2N^2$ and the correction to the classical recursion relation is suppressed by a factor of n^2/N. As we compute the norm using the recursion relation we accumulate errors yielding an error in the norm of $|n\rangle$ of the order of

$$\frac{n^3}{N\,.} \qquad (6.19)$$

We have seen this number before. It is the momentum expansion of M-theory. The size of the S^4 is $R\sim l_pN^{1/3}$ using this relation the expression (6.19) becomes

$$\left(\frac{P}{M_p}\right)^3.$$

This is to be expected since for modes for which low energy supergravity is legitimate we should see only small deviations from the behavior of wave functions on a fixed sphere.

Next we would like to evaluate the expectation value B^2 as seen by the n-th spherical harmonics

$$\frac{1}{4}J_{AA'}J_{BB'}B^{AB}B^{A'B'}|n\rangle\,.$$

We will write it as the sum of two terms:

$$\begin{aligned}
B^{12}B^{12}+\frac{1}{2}(B^{14}B^{32}+B^{32}B^{14})|n\rangle &= 4(N-n)(N-n-1)\frac{N!2^n}{(N-n)!}\times \\
&\quad\times\epsilon^{\cdots}\epsilon^{\cdots}{B^{13}_{\dots}}^{n}({B^{12}_{\dots}}^{2}+B^{14}_{\dots}B^{32}_{\dots}){B^{0}_{\dots}}^{N-n-2}|+\rangle + \\
&\quad+3(N-n)|n\rangle\frac{1}{2}(B^{13}B^{24}+B^{24}B^{13})|n\rangle \\
&= 4(N-n)(N-n-1)\times \\
&\quad\times\epsilon^{\cdots}\epsilon^{\cdots}{B^{13}_{\dots}}^{n+1}B^{24}_{\dots}{B^{0}_{\dots}}^{N-n-2}|+\rangle + \qquad (6.20)\\
&\quad+((N-n)+(n+1)(N-n)+n(N-n+1))|n\rangle\,.
\end{aligned}$$

The radius of the sphere is therefore given by

$$2(N-n)(N-n-1)+(N-n)+n(N-n+1)+(n+1)(N-n) = 2N^2\left(1+O\left(\frac{n}{N}\right)\right).$$

6.5 The definition of U

We have seen that on the relevant states

$$\sum_i B^{i2} \sim 2N^2 \,,$$

and what are the corrections to this equation. The full definition of the 5 coordinates

$$Y^{AB} \sim \frac{B^{AB}}{Q^2}$$

implies

$$\sum_i Y^{i2} \sim \frac{2N^2}{Q^4} \,. \tag{6.21}$$

We could leave the matter at that, i.e. have 5 non-commuting variables, but since spacetime factorizes into a sphere and the radial coordinate which become part of the AdS_7 we will use the B operators as constrained coordinates on S^4 and define an additional commuting coordinate $\bar{U}$ by

$$\bar{U} = \frac{\sqrt{2}N}{Q^2} \,. \tag{6.22}$$

The advantage of this definitions is that all the non-commutativity is associated with parts of space which carry the flux, i.e. the S^4, whereas the AdS_7 is a regular commutative sigma model. This enforces the claim that in some cases there is a fuzziness in spacetime associated with field non-zero field strengths.

6.6 Mapping of supercharges

We have seen that there is a natural identification of states on the Higgs branch as states on the near-horizon Coulomb branch. We would now like to begin addressing the issue of dynamics, i.e. can we see that the dynamics on the Higgs branch is mapped to the expected dynamics on the Coulomb branch. To do so we would like to show how the supercharges of the two systems are related. In this section we will partially show how the supercharges on the Coulomb branch are mapped onto the supercharges of the reduced model, after we rewrite the Coulomb branch coordinates in term of Higgs branch variables. In the process we will also need to discuss the transformation laws of the remaining fermions from the near-horizon Coulomb branch (where we denoted them by ρ) to the Higgs branch (where we denoted them by η).

The supersymmetry generators along the Coulomb (5.3) branch contain the term

$$U^{3/2} \rho^{\alpha}_{B} \gamma^{i}_{CA} J^{BC} \frac{\partial}{\partial U^i} \tag{6.23}$$

(we will neglect R, M_p and N_4 dependence). We will show that under the map (6.1) this term becomes some of the terms in the supercharges of the Higgs branch.

Replacing the commuting coordinate U by the non-commutative definition (6.1) is ambiguous because of ordering issues. The difference should be terms suppressed by powers of $1/N$ and therefore beyond the scope of the supercharges (5.3) anyhow (which are just the leading (in derivatives) terms in the lagrangean). More practically, we will manipulate the U variables as commuting variables in the Coulomb branch supercharges for as long as possible.

Inserting

$$\frac{\partial}{\partial U^j} = \frac{1}{U^2}\left(U^i L'^{ij} + U^j \left(U^i \frac{\partial}{\partial U^i}\right)\right),$$

where L' are SO(5) generators $U^i\partial_{U^j} - U^j\partial_{U^i}$. Defining new fermion operators

$$\eta^\alpha_A = \rho^\alpha_B \gamma^i_{CA} J^{BC} \frac{U^i}{U}, \qquad \rho^\alpha_B = \frac{1}{2}\eta^\alpha_B J^{BC}\gamma^i_{CA}\frac{U^i}{U} \tag{6.24}$$

where we have used the Fierz relation:

$$\gamma^i {\gamma^j}^* + \gamma^j {\gamma^i}^* = -2\delta^{ij}, \qquad {\gamma_j}^* = -J\gamma^j J,$$

η will be the fermion fields on the Higgs branch.

Using this definition of ρ as a function of η we obtain that this part of the hamiltonian is

$$\eta^\alpha_A U^{1/2}(U^i\partial_{U^i}) + \frac{1}{2}U^{-3/2}\eta^\alpha_{B'} J^{B'D}\gamma^j_{DB} J^{BC}\gamma^l_{CA} U^l U^i L'^{ij}.$$

We can now transcribe this expression to the Higgs branch variables:

1. Using the relation $U \sim 1/Q^2$, the first term becomes

$$\eta^\alpha_A \partial_Q,$$

   which is one of the terms on the Higgs branch.

2. The 2nd expression can be simplified further. The expression $\gamma^j_{DB}J^{BC}\gamma^l_{CA}$ can give us two kinds of terms with different quantum numbers. One which is proportional to $\delta^{ij}J_{DA}$ which is zero because $U^lU^iL'^{ij}\delta^{jl} = 0$ and another which is proportional to γ^{ij}_{DA}. The latter is multiplied by

$$U^lU^iL'^{ij} - U^jU^iL'^{il},$$

   where we have anti-symmetrized on $l \leftrightarrow j$. This, however, equals $U^2L'^{jl}$ on all spherical harmonics. The key to the transcription now is to identify the SO(5) generators L' on the coordinates of S^4 as the SO(5) generators L on the ν-fermion (which is the full SO(5) generator on the reduced system). Furthermore, we write U^2 in terms of Q, and obtain precisely the ηL term in (4.11).

7. Discussion

We have seen that for quantum theories that are obtained as a decoupled Higgs branch there is a natural change of variables to the Coulomb branch. When these theories are viewed as DLCQ descriptions of decoupled theories in M/String theory, then this change of variables is nothing but the AdS/CFT correspondence (and its generalizations to non-conformal cases). This is not surprising since the Higgs branch quantum mechanics is believed to be the full DLCQ of these theories, but what is a pleasant surprise is that this change of variables, and hence the AdS/CFT correspondence, is fairly straightforward in terms of the decoupling process.

Since the full DLCQ is valid in all cases, even large curvature or large higher genus corrections, we can use it to explore gravity in these cases. The most striking feature that we have found in this paper is that in the case of $AdS_7 \times S^4$ the 4-sphere becomes a non-commutative space (we have shown this for the $N_0 = 1$ case but we expect it to be true more generally). It is known that open strings generate a non-commutative structure when turning on field strengths on D-branes[7] [26], but the status of turning on field strengths in the closed string/M-theory sector (which in our case is a flux on S^4) was less clear. Our results suggest that in this case also a similar fuzziness of space appears.

Acknowledgments

We would like to thank O. Aharony, A. Kapustin, M. Rozali and N. Seiberg for useful discussions. The work of MB is supported by NSF grant PHY-9513835, and of H.V. by NSF-grant 98-02484, a Pionier fellowship of NWO, and the Packard foundation.

A. A toy model

In this appendix we discuss a particularly simple case of reducing a model. This will serve to explain which operations are allowed in the reduced model and which are not.

The toy model contains 3 real bosonic coordinates and 3 complex fermionic coordinates $X^{1,2,3}$, $\psi^{1,2,3}$, $\bar{\psi}^{1,2,3}$ and is invariant under the obvious SO(3):

$$Q = \psi^i \partial_i \,, \qquad \bar{Q} = \bar{\psi}^i \partial_i \,.$$

The SO(3) generators are

$$M^{ij} = X^i \partial_j - X^j \partial_i + \psi^i \bar{\psi}^j - \psi^j \bar{\psi}^i \,.$$

[7] One usually turns on a B field in the bulk but we can gauge it to a field strength on the brane.

We are interested in reducing the model with respect SO(3). The wave functions satisfy $M^{ij}\Psi = 0$. This implies that for $j = 2, 3$:

$$\partial_j = \frac{1}{X^1}\left(X^j\partial_1 - \psi^1\bar{\psi}^j + \psi^j\bar{\psi}^1\right).$$

Since we will insert this expression every time there will be a derivative ∂_j we can take the limit $X^2, X^3 = 0$ without worry. (Note for example that we can not use this formula to compute, for example, $[\partial_i, X^j]$ because X^j is not SO(3) invariant. But we can compute things like $[\partial_j, X^2] = 2X^j$).

This gives us supercharges

$$\begin{aligned} Q &= \psi^1\partial_1 - \psi^j\frac{1}{X^1}\left(\psi^1\bar{\psi}^j - \psi^j\bar{\psi}^1\right) = \psi^1\left(\partial_1 + \psi^j\bar{\psi}^j\right), \\ \bar{Q} &= \psi^1\partial_1 - \bar{\psi}^j\frac{1}{X^1}\left(\psi^1\bar{\psi}^j - \psi^j\bar{\psi}^1\right) = \bar{\psi}^1\left(\partial_1 - \psi^j\bar{\psi}^j\right). \end{aligned} \tag{A.1}$$

It is easy to verify that $\{Q, Q\} = 0$. To evaluate the hamiltonian we anti-commute $\{Q, \bar{Q}\}$. Even though we started with an hamiltonian that did not contain fermion oscillators, the new hamiltonian does. Still, the change is only the generation of new 3-fermion terms in the supercharges.

B. Computation of the supercharges

In this appendix we will (partially) determine the supercharges of the reduced Higgs branch quantum mechanics. One can compute the supercharges of the full ADHM sigma model and then compute the reduced supercharges. Alternatively, since the supercharges are determined by the (super)symmetries of the problem, one can compute them in the reduced model directly, which is the route we will take. We will show that the supercharges are

$$Q^\alpha_A = \eta^\alpha_A\Big(\frac{\partial}{\partial Q} + \frac{a}{Q}\Big) - \frac{2}{Q}\eta^\alpha_B L^{BC}J_{AC} + \frac{1}{Q}M^{\alpha\beta}\eta^\gamma_A\epsilon_{\beta\gamma}, \tag{B.1}$$

where a is a specific number which we have not determined. The different operators in (B.1) are the following:

1. $M^{\alpha\beta} = J^{BB'}\eta^\alpha_B\eta^\beta_{B'}$ denote the generators of $\mathrm{SU}(2)_R$ in the η sector. These operators are symmetric under the exchange $\alpha \leftrightarrow \beta$ and satisfy

$$\begin{aligned} [M^{\alpha\beta}, M^{\alpha'\beta'}] &= \epsilon^{\alpha\alpha'}M^{\beta\beta'} + \epsilon^{\alpha\beta'}M^{\beta\alpha'} + \epsilon^{\beta\alpha'}M^{\alpha\beta'} + \epsilon^{\beta\beta'}M^{\alpha\alpha'} \\ [M^{\alpha\beta}, \eta^\gamma_A] &= \epsilon^{\alpha\gamma}\eta^\beta_A + \epsilon^{\beta\gamma}\eta^\alpha_A. \end{aligned} \tag{B.2}$$

2. $L^{AB} = \nu^A_k\bar{\nu}^k_C J^{BC} + \nu^B_k\bar{\nu}^k_C J^{AC}$ are the generators of USp(2) on the ν sector. They satisfy

$$[L^{AB}, L^{A'B'}] = J^{AA'}L^{BB'} + J^{AB'}L^{BA'} + J^{BA'}L^{AB'} + J^{BB'}L^{AA'}.$$

As usual in the hamiltonian formalism one first writes down the most general supercharge and then fixes it by requiring closure on the hamiltonian. The most general supercharge (setting the coefficient of derivative term to 1) is given by

$$\eta^\alpha_A\Big(\frac{\partial}{\partial Q}+\frac{a}{Q}\Big)+\frac{c_1}{Q}\eta^\alpha_B\nu^B_k\bar\nu^k_A+\frac{c_2}{Q}\eta^\alpha_B\nu^C_k\bar\nu^k_D J^{BD}J_{AC}+\frac{b}{Q}M^{\alpha\beta}\eta^\gamma_A\epsilon_{\beta\gamma}.$$

This hamiltonian is determined in the following way

1. The powers of Q in front of each term are determined by scale invariance under which the fermion operators have dimension 0.
2. One might think to add a term $\frac{1}{Q}\eta^\alpha_A\nu^D_k\bar\nu^k_D$, but as explained above, this number is a constant in the relevant sector and therefore can be absorbed into a.
3. Otherwise one uses $\mathrm{SU}(2)_R\times\mathrm{USp}(2)\times\mathrm{SU}(N)$ symmetry where the last component is the remaining flavor symmetry after reducing the model.

We will be interested in computing the 4-Fermi term in the anti-commutator of two supercharges, and require that the result is a singlet under all the global symmetries. This will determine[8] c_1, c_2 and b. The 4-Fermi terms that we might generate are $\eta\eta\eta\eta$, $\nu\bar\nu\nu\bar\nu$ and $\eta\eta\nu\bar\nu$.

Since we will be interested in 4-Fermi terms we can anti-commute fermion freely, neglecting the C-number that results, since it will appear as part of the 2-Fermi term.

The $\eta\eta\eta\eta$ term. The relevant terms that contribute to the 4-η term are

$$\begin{aligned}&\Big\{\eta^\alpha_A\partial_Q,\frac{b}{Q}M^{\alpha'\beta'}\eta^{\gamma'}_{A'}\epsilon_{\beta'\gamma'}\Big\}+\Big\{\frac{b}{Q}M^{\alpha\beta}\eta^\gamma_A\epsilon_{\beta\gamma},\eta^{\alpha'}_{A'}\partial_Q\Big\}+\Big\{\frac{b}{Q}M^{\alpha\beta}\eta^\gamma_A\epsilon_{\beta\gamma},\frac{b}{Q}M^{\alpha'\beta'}\eta^{\gamma'}_{A'}\epsilon_{\beta'\gamma'}\Big\}\cong\\ &\qquad\cong\frac{-b}{Q^2}\Big(\eta^\alpha_A M^{\alpha'\beta'}\eta^{\gamma'}_{A'}\epsilon_{\beta'\gamma'}+((A,\alpha)\leftrightarrow(A',\alpha'))\Big)+\\ &\qquad\quad+\frac{b^2}{Q^2}\Big(\epsilon^{\alpha\alpha'}\epsilon_{\beta\gamma}\epsilon_{\beta'\gamma'}M^{\beta\beta'}\eta^\gamma_A\eta^{ga'}_{A'}+M^{\alpha\alpha'}(\epsilon_{\gamma\gamma'}\eta^\gamma_A\eta^{\gamma'}_{A'})\Big)\,.\end{aligned}\tag{B.3}$$

Where $\cong$ denotes that we have neglected terms that are already singlets under $\mathrm{SU}(2)_R\times\mathrm{USp}(2)$.

This expression is an $\mathrm{SU}(2)_R$ singlet, in which case it is either in the **1** or **5** of $\mathrm{USp}(2)$, and an $\mathrm{SU}(2)_R$ triplet, in which case it is in the **10** of $\mathrm{USp}(2)$. We require the cancelation of the $(\mathbf{1},\mathbf{5})$ and $(\mathbf{3},\mathbf{10})$ terms.

To extract the $(\mathbf{1},\mathbf{5})$ terms we contract the expression with $\epsilon_{\alpha\alpha'}$ and obtain

$$\begin{aligned}&-\frac{2b}{Q^2}\Big(-\epsilon_{\delta\delta'}\epsilon_{\beta'\gamma'}M^{\delta'\beta'}\eta^\delta_A\eta^{\gamma'}_{A'}\Big)\\ &-\frac{2b^2}{Q^2}\epsilon_{\beta\gamma}\epsilon_{\beta'\gamma'}M^{\beta\beta'}\eta^\gamma_A\eta^{\gamma'}_{A'}\,.\end{aligned}$$

[8] We have not calculated the 2-Fermi term, hence we can not compute a. The system of equations for the other coefficient is over-determined

For this term to cancel we require

$$b = 1\,.$$

The solution $b = 0$ is of course also a valid solution, however, one easily sees that this can not be the right solution — already at $N_4 = 2$ we have a 3-η term in the supercharges.

To extract the $(\mathbf{3}, \mathbf{10})$ term we symmetrize $A \leftrightarrow A'$ and obtain

$$-\frac{b}{Q^2} M^{\alpha'\alpha} \epsilon_{\gamma\gamma'} \eta_A^\gamma \eta_{A'}^{\gamma'} + \frac{b^2}{Q^2} M^{\alpha\alpha'} \epsilon_{\gamma\gamma'} \eta_A^\gamma \eta_{A'}^{\gamma'}\,,$$

which again gives us $b = 1$.

The $\nu\bar\nu\nu\bar\nu$ term. The anti-commutator that contributes to the 4-ν is

$$\left\{ \frac{c_1}{Q} \eta_B^\alpha \nu_k^B \bar\nu_A^k + \frac{c_2}{Q} \eta_B^\alpha \nu_k^C \bar\nu_D^k J^{BD} J_{AC}, \frac{c_1}{Q} \eta_{B'}^{\alpha'} \nu_{k'}^{B'} \bar\nu_{A'}^{k'} + \frac{c_2}{Q} \eta_{B'}^{\alpha'} \nu_{k'}^{C'} \bar\nu_{D'}^{k'} J^{B'D'} J_{A'C'} \right\}$$

It is clear that the expression is symmetric to the exchange $(\alpha, A) \leftrightarrow (\alpha', A')$ and it is also clear that the result will be proportional to a singlet of $\mathrm{SU}(2)_R$, i.e. to $\epsilon^{\alpha\alpha'}$. Hence the result will antisymmetric under $A \leftrightarrow A'$. Under USp(2) the expression is either a $\mathbf{1}$ or a $\mathbf{5}$.

This anticommutator is

$$\frac{c_1^2}{Q^2}\Big(J_{BB'} (\nu_k^B \bar\nu_A^k \nu_{k'}^{B'} \bar\nu_{A'}^{k'}) \Big) + \frac{c_1 c_2}{Q^2} \Big(J_{A'C'} \nu_k^B \bar\nu_A^k \nu_{k'}^{C'} \bar\nu_B^{k'} - J_{AC'} \nu_k^B \bar\nu_{A'}^k \nu_{k'}^{C'} \bar\nu_B^{k'} \Big) + \\ + \frac{c_2^2}{Q^2} \Big(J_{AC} J_{A'C'} \nu_k^C \bar\nu_D^k \nu_{k'}^{C'} \bar\nu_B^{k'} J^{BD} \Big)\,. \qquad \text{(B.4)}$$

We would like to ask what are the conditions on c_1 and c_2 such that this expression is a singlet of USp(2). To do this, its enough to examine special cases. for example $A = 1, A' = 3$. In this case we obtain the expression $(J^{12} = J^{34} = 1)$

$$-\frac{c_1^2}{Q^2} \left(J_{BB'} \nu_k^B \nu_{k'}^{B'} \right) \bar\nu_1^k \bar\nu_3^{k'} - \frac{c_1 c_2}{Q^2} \left(\nu_k^B \bar\nu_B^{k'} \right) \left(\bar\nu_1^k \nu_{k'}^4 - \bar\nu_3^k \nu_{k'}^2 \right) + \frac{c_2^2}{Q^2} \left(\bar\nu_D^k \bar\nu_B^{k'} J^{BD} \right) \nu_k^2 \nu_{k'}^4\,.$$

Each term in this expression contain either $\nu^2 \bar\nu_3$ or $\nu^4 \bar\nu_1$. To show under what conditions its zero its enough to focus on terms containing one of this 2-fermion elements, say $\nu^2 \bar\nu^3$. The terms containing the 2-fermion element are

$$-\frac{1}{Q^2} \left(c_1^2 \nu_k^1 \bar\nu_1^k + c_2^2 \nu_k^4 \bar\nu_4^k \right) \nu_{k'}^2 \bar\nu_3^{k'} - \frac{1}{Q^2} \left(c_1^2 \nu_{k'}^1 \bar\nu_1^k + c_2^2 \nu_{k'}^4 \bar\nu_4^k \right) \nu_k^2 \bar\nu_3^{k'} - \frac{c_1 c_2}{Q^2} \left(\nu_k^B \bar\nu_B^{k'} \right) \bar\nu_3^k \nu_{k'}^2\,.$$

There are several ways for this to be zero. The simple one is to set $c_1 + c_2 = 0$. In this case the expression vanishes. This is however not the only way. Another way is to set $c_1 = c_2$. The reason is that we do not really need to require that the expression vanishes identically, rather it should vanish only on the flavor invariant gauge invariant states in our Hilbert space. As explained in section 4.2 these states are of the form

$$\epsilon^{l_1\cdots l_N}\epsilon^{t_1\cdots t_N}\nu_{l_1}^{A_1}\cdots\nu_{l_N}^{A_N}\nu_{t_1}^{B_1}\cdots\nu_{t_N}^{B_N}\left|\phi\right\rangle . \tag{B.5}$$

On these states $\nu_k^B\nu_B^{k'} \propto \delta_k^{k'}$ (with a fixed coefficient). Therefore terms containing $\nu_k^B\nu_B^{k'}$ are not really 4-Fermi terms. Hence the term which is proportional to c_1c_2 does not contribute. Similarly we can substitute (setting $c_1 = c_2$)

$$\nu_{k'}^1\bar\nu_1^k + \nu_{k'}^4\bar\nu_4^k = -\nu_{k'}^2\bar\nu_2^k + \nu_{k'}^3\bar\nu_3^k + \text{fixed number}$$

to obtain that the total expression is

$$-\frac{c^2}{Q^2}(\nu_{k'}^B\bar\nu_B^k)\nu_{k'}^2\bar\nu_3^{k'} ,$$

which is again a 2-Fermi operator on the relevant Hilbert space.

The solution which is correct is actually $c_1 = c_2$. In this case, the two $\eta\nu\bar\nu$ terms combine to give

$$c\eta_B^\alpha L^{BC}J_{AC} .$$

Be before we proceed it is worth returning to the $\mathrm{USp}(2)\times\mathrm{SU}(2)_R$ singlet terms which we have been neglecting left and right. These terms do not give us any restrictions on the supercharges but they are part of the hamiltonian. The $\nu^2\bar\nu^2$ term in the hamiltonian turns out to be proportional to

$$J^{AA'}J_{BB'}(\nu_k^B\bar\nu_A^k\nu_{k'}^{B'}\bar\nu_{A'}^{k'}) ,$$

which is in turn proportional to the USp(2) 2nd Casimir (up to 2 Fermi terms which we have not calculated).

The $\eta\eta\nu\bar\nu$ term. The anti-commutators which can contribute to such a term are:

$$\begin{aligned}
&\text{I.} && \left\{\eta_A^\alpha\partial_Q, \frac{c}{Q}\eta_{B'}^{\alpha'}L(\nu)^{B'C'}J_{C'A'}\right\} + (A,\alpha)\leftrightarrow(A',\alpha') ,\\
&\text{II.} && \left\{\frac{c}{Q}\eta_B^\alpha L(\nu)^{BC}J_{AC}, \frac{c}{Q}\eta_{B'}^{\alpha'}L(\nu)^{B'C'}J_{A'C'}\right\} ,\\
&\text{III.} && \left\{\frac{c}{Q}\eta_B^\alpha L(\nu)^{BC}J_{AC}, \frac{1}{Q}M^{\alpha'\beta'}\eta_{A'}^{\gamma'}\epsilon_{\beta'\gamma'}\right\} + (A,\alpha)\leftrightarrow(A',\alpha') .
\end{aligned}$$

Evaluating these terms we obtain

$$\begin{aligned}
&\text{I.} && \frac{-c}{Q^2}\left(\eta_A^\alpha\eta_B^{\alpha'}L^{BC}J_{A'C} + \eta_{A'}^{\alpha'}\eta_B^\alpha L^{BC}J_{AC}\right) ,\\
&\text{II.} && \frac{c^2}{Q^2}\Big(-J^{B'B}\eta_B^\alpha\eta_{B'}^{\alpha'}L^{CC'}J_{AC}J_{A'C'} + \eta_B^\alpha\eta_{B'}^{\alpha'}L^{BB'}J_{A'A} -\\
& && \quad - \eta_{A'}^\alpha\eta_{B'}^{\alpha'}L^{CB'}J_{AC} + \eta_B^\alpha\eta_A^{\alpha'}L^{BC'}J_{A'C'}\Big) ,\\
&\text{III.} && \frac{bc}{Q^2}\left(\epsilon^{\alpha\alpha'}\eta_B^{\beta'}\eta_{A'}^{\gamma'}\epsilon_{\beta'\gamma'}L^{BC}J_{AC} + \eta_B^{\alpha'}\eta_{A'}^\alpha L^{BC}J_{AC} - M^{\alpha'\alpha}J_{BA'}L^{BC}J_{AC}\right) +\\
& && +(\alpha,A)\leftrightarrow(\alpha',A') .
\end{aligned}$$

The $(\mathbf{1},\mathbf{5})$ under SU(2) × USp(2) part is:

$$\frac{1}{Q^2}(2c+c^2)\left(\eta^{\alpha}_{A}\epsilon_{\alpha\alpha'}\eta^{\alpha'}_{B}L^{BC}J_{A'C}-\eta^{\alpha}_{A'}\epsilon_{\alpha\alpha'}\eta^{\alpha'}_{B}L^{BC}J_{AC}\right)$$

giving us $c=-2$.

To examine the $(\mathbf{3},\mathbf{10})$ part we write the expression for $\alpha=\alpha'=0$, we then get:

$$\frac{1}{Q^2}(-2c-c^2)\left(\eta^{0}_{A}\eta^{0}_{B}L^{BC}J_{A'C}+\eta^{0}_{A'}\eta^{0}_{B}L^{BC}J_{AC}\right)$$

giving us again $c=-2$.

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[2] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[3] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of* (2,0) *superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [hep-th/9712117].

[4] E. Witten, *On the conformal field theory of the Higgs branch*, *J. High Energy Phys.* **07** (1997) 003 [hep-th/9707093].

[5] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg and E. Silverstein, *Matrix description of interacting theories in six dimensions*, *Adv. Theor. Math. Phys.* **1** (1998) 148 [hep-th/9707079].

[6] M. Berkooz and M.R. Douglas, *Five-branes in m(atrix) theory*, *Phys. Lett.* **B 395** (1997) 196 [hep-th/9610236].

[7] A. Jevicki and S. Ramgoolam, *Non-commutative gravity from the AdS/CFT correspondence*, *J. High Energy Phys.* **04** (1999) 032 [hep-th/9902059].

[8] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N=4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[9] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[10] H. Awata and S. Hirano, *AdS_7/CFT_6 correspondence and matrix models of m5-branes*, *Adv. Theor. Math. Phys.* **3** (1999) 147 [hep-th/9812218].

JHEP11(1999)037

[11] R. Dijkgraaf, E. Verlinde and H. Verlinde, *BPS spectrum of the five-brane and black hole entropy*, *Nucl. Phys.* **B 486** (1997) 77 [hep-th/9603126]; *BPS quantization of the five-brane*, *Nucl. Phys.* **B 486** (1997) 89 [hep-th/9604055];
R. Dijkgraaf, E. Verlinde and H. Verlinde, *5d black holes and matrix strings*, *Nucl. Phys.* **B 506** (1997) 121 [hep-th/9704018].

[12] M. Berkooz, M. Rozali and N. Seiberg, *On transverse fivebranes in m(atrix) theory on T^4 and T^5*, *Phys. Lett.* **B 408** (1997) 105 [hep-th/9704089].

[13] N. Seiberg, *New theories in six dimensions and matrix description of m- theory on T^5 and T^5/Z_2*, *Phys. Lett.* **B 408** (1997) 98 [hep-th/9705221].

[14] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS 5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[15] J. Curtis G. Callan, J.A. Harvey and A. Strominger, *Supersymmetric string solitons*, in Trieste 1991, Proceedings, String Theory and Quantum Gravity, 1991, p. 208, [hep-th/9112030].

[16] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025;
T. Banks and N. Seiberg, *Strings from matrices*, *Nucl. Phys.* **B 497** (1997) 41 [hep-th/9702187];
R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[17] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *J. High Energy Phys.* **04** (1999) 017 [hep-th/9903224].

[18] O.J. Ganor and S. Sethi, *New perspectives on Yang-Mills theories with sixteen supercharges*, *J. High Energy Phys.* **01** (1998) 007 [hep-th/9712071]:
A. Kapustin and S. Sethi, *The Higgs branch of impurity theories*, *Adv. Theor. Math. Phys.* **2** (1998) 571 [hep-th/9804027].

[19] In preparation

[20] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $\mathcal{N} = 2$ supersymmetric gauge theory*, *Phys. Rev.* **D 54** (1996) 2921 [hep-th/9603136];
V.V. Khoze, M.P. Mattis and M.J. Slater, *The instanton hunter's guide to supersymmetric* U(N) *gauge theories*, *Nucl. Phys.* **B 536** (1998) 69 [hep-th/9804009].

[21] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[22] D. Marolf and A. Peet, *Brane baldness vs. superselection sectors*, *Phys. Rev.* **D 60** (1999) 105007 [hep-th/9903213].

[23] S. Sethi and M. Stern, *D-brane bound states redux*, *Comm. Math. Phys.* **194** (1998) 675 [hep-th/9705046].

[24] H. Nakajima, *Resolution of moduli spaces of ideal instantons on* $\mathbb{R}^4$, in *Topology, geometry and field theory*, Fukaya, Furuta, Kohno and Kotschick eds., World Scientific, Singapore 1994.

[25] D.-E. Diaconescu and R. Entin, *A non-renormalization theorem for the* $d = 1$, $N = 8$ *vector multiplet*, *Phys. Rev.* **D 56** (1997) 8045 [`hep-th/9706059`].

[26] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [`hep-th/9711162`]; M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [`hep-th/9711165`].